

A unified approach to entanglement criteria using the Cauchy-Schwarz and Hölder inequalities

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(Dated: May 6, 2014)

We present unified approach to different recent entanglement criteria. Although they were developed in different ways, we show that they are all applications of a more general principle given by the Cauchy-Schwarz inequality. We explain this general principle and show how to derive with it not only already known but also new entanglement criteria. We systematically investigate its potential and limits to detect bipartite and multipartite entanglement.

PACS numbers: 03.67.-a, 03.65.Ud

I. INTRODUCTION

The phenomenon of entanglement is of fundamental interest since it is a main difference between the classical and the quantum world. Furthermore, it is believed to be the central resource for quantum computing and protecting quantum communication. The existing entanglement criteria solve the problem of characterizing entanglement only for certain classes of states and, in addition, some of them are very resource intensive if applied experimentally. For example, to apply the famous positive-partial-transpose (PPT) criterion [1, 2] experimentally, a full quantum state tomography is necessary in practice. Some other entanglement criteria are formulated as inequalities for mean values of observables [3–8]. This sort of entanglement criteria are especially useful to detect entanglement in experiments since complete knowledge about the quantum state is not necessary.

When going from bipartite entanglement to multipartite entanglement, the detection and characterization of entanglement become even more complicated. First of all, there exist different degrees of entanglement. That is, an N -partite entangled state ϱ may be a convex combination of pure entangled states with maximally k entangled parties. If at least one N -partite entangled pure state is necessary to form ϱ , we call the state genuine multipartite entangled. Since the representation of a mixed state by a convex sum of pure states is not unique, the entanglement characterization of multipartite states is more than the combination of bipartite entanglement criteria. For example, there exist states which are entangled under every bipartite split but are not genuine multipartite entangled. On the other hand, there exist states, which are separable under every possible bipartite split, but not fully separable.

Consequently, there are many different criteria and there are many different ways to develop them. To give an example, Hillery and Zubairy developed in Ref. [4] first a criterion based on uncertainties, followed by a generalized entanglement criterion solely based on the Cauchy-Schwarz inequality and the properties of separable states.

Whereas this approach used only a single operator per subsystems, we gain more freedom by using two operators per subsystem. In this way, we develop in this paper a general principle to create entanglement criteria. Many already existing criteria follow immediately from this new principle, but also new criteria can be created.

This paper consists of two main parts: In Sec. II we introduce our scheme of developing entanglement criteria with the help of the Cauchy-Schwarz inequality. Here, we will first concentrate on bipartite entanglement in Sec. II A. We explain the intimate connection of the PPT criterion [1, 2] to our criterion and give explicit examples of classes of states which can and which can not be detected by our criterion. Afterwards, in Sec. II B, we generalize our scheme to multipartite entanglement by using the Hölder inequality and show that in the multipartite case our criterion can detect entangled states which cannot be detected by the PPT criterion. In Sec. III we discuss several already existing criteria and demonstrate how they can be derived within our scheme. In this way, a connection between the different criteria are pointed out and the advantages and disadvantages of the different criteria become visible.

II. ENTANGLEMENT CRITERIA FROM THE CAUCHY-SCHWARZ INEQUALITY

First, recall that a two-particle state ϱ is separable, if it can be written as a mixture of product states

$$\varrho = \sum_k p_k \varrho_k^A \otimes \varrho_k^B, \quad (1)$$

where the p_k form a probability distribution. We aim at deriving entanglement criteria in the form of inequalities which hold for separable states, but which can be violated by entangled states. Our main tool are two simple facts: (i) we use the property of product states $\varrho = \varrho^A \otimes \varrho^B$ that

$$\langle AB \rangle_{\text{ps}} = \langle A \rangle_{\varrho^A} \langle B \rangle_{\varrho^B} \quad (2)$$

for operators A and B acting on Alice's and Bobs subsystem, respectively. Here and in the following, $\langle \dots \rangle_{\text{ps}}$ denotes that the expectation value is taken for a product state. Similarly $\langle \dots \rangle_{\text{sep}}$ denotes an expectation value for a separable state.

(ii) The Cauchy Schwarz (CS) inequality [9]

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle \quad (3)$$

where x and y are two vectors and $\langle \cdot, \cdot \rangle$ defines an inner product.

A. Bipartite entanglement

We concentrate on expectation values of a bipartite system which can be written as $\langle A_1 A_2 B_1 B_2 \rangle$ where the operators A_i and B_j acting on Alice's and Bob's subsystem, respectively. For a pure state, these expectation values can be interpreted as the inner product of the two vectors $|\psi_1\rangle \equiv A_1^\dagger B_1^\dagger |\psi\rangle$ and $|\psi_2\rangle \equiv A_2 B_2 |\psi\rangle$. Since the scalar product is bilinear, all fragmentation of a scalar product into a bra- and a ket-vector can be described by two operators per subsystems. More than two are not necessary, since they can be always combined to a single operator acting on the bra- and one acting on the ket-vector. The expectation value $\langle A_1 A_2 B_1 B_2 \rangle$ can be used to detect entanglement by using the following theorem:

Theorem II.1 *The inequality*

$$|\langle A_1 A_2 B_1 B_2 \rangle_{\text{sep}}|^2 \leq \langle A_1 A_1^\dagger B_2^\dagger B_2 \rangle_{\text{sep}} \langle A_2^\dagger A_2 B_1 B_1^\dagger \rangle_{\text{sep}}. \quad (4)$$

is valid for separable states and can only be violated by entangled states.

Proof For product states $|\phi\rangle = |a\rangle|b\rangle = |a, b\rangle$, we can write the expectation value as the product of the expectation values of the subsystems. By applying the CS inequality to every single subsystem we get

$$\begin{aligned} |\langle a, b | A_1 A_2 B_1 B_2 | a, b \rangle|^2 &= |\langle a | A_1 A_2 | a \rangle|^2 |\langle b | B_1 B_2 | b \rangle|^2 \\ &\leq \langle a | A_1 A_1^\dagger | a \rangle \langle a | A_2^\dagger A_2 | a \rangle \\ &\quad \times \langle b | B_1 B_1^\dagger | b \rangle \langle b | B_2^\dagger B_2 | b \rangle \end{aligned} \quad (5)$$

which is equal to

$$|\langle A_1 A_2 B_1 B_2 \rangle_{\text{ps}}|^2 \leq \langle A_1 A_1^\dagger B_2^\dagger B_2 \rangle_{\text{ps}} \langle A_2^\dagger A_2 B_1 B_1^\dagger \rangle_{\text{ps}}. \quad (6)$$

In order to see that this inequality holds also for mixed states, note first that the root of the left-hand side $|\langle A_1 A_2 B_1 B_2 \rangle_{\text{ps}}|$ is convex in the state, while the root of the right-hand side is of the type $\sqrt{f(\varrho)g(\varrho)}$, where f and g are positive functions. This implies that it is concave [5]. Therefore, this inequality is also valid for mixtures of product states which proves the correctness of Eq. (4) for any separable state.

With the help of the CS inequality we derive for general states

$$|\langle A_1 A_2 B_1 B_2 \rangle|^2 \leq \langle A_1 A_1^\dagger B_1 B_1^\dagger \rangle \langle A_2^\dagger A_2 B_2^\dagger B_2 \rangle, \quad (7)$$

which provides a tight upper limit for all states.

Depending on the choice of the operators A_j and B_j the inequality Eq. (4) may provide a stricter limit than Eq. (7). If yes, then there exist states which violates Eq. (4). These states must be entangled and therefore Eq. (4) is able to detect entanglement.

Regarding the application of Theorem II.1 we note: (i) The ability of Eq. (4) to detect entanglement depends on the chosen operators A_j and B_j . For example, by a comparison of Eq. (7) and Eq. (4) we immediately find that if $A_1 A_1^\dagger = A_2^\dagger A_2$ or $B_1 B_1^\dagger = B_2^\dagger B_2$ Eq. (4) cannot be violated for any state. (ii) The optimal choice of the operators A_j and B_j depends in general on the state ϱ . (iii) Whereas Eq. (4) is valid for general mixed separable states, Eq. (5) used to develop Eq. (4) is only valid for pure product states. As a consequence, great care concerning convexity has to be taken when generalizing Eq. (4) to multipartite systems.

After the development of the entanglement criteria Eq. (4) we will investigate the bipartite case now in more detail. First, we discuss the best choice of operators.

Theorem II.2 *The best choice of the operators A_j are given by*

$$A_1 = |a\rangle\langle\varphi|; \quad A_2 = |\varphi\rangle\langle\alpha| \quad (8)$$

with $|a\rangle$, $|\alpha\rangle$ and $|\varphi\rangle$ being pure states of Alice's subsystem, and an analogous choice for Bob.

Proof Any pair of operators A_1 and A_2 can be written as

$$A_1 = \sum_j x_j |a_j\rangle\langle\varphi_j| \quad (9)$$

$$A_2 = \sum_j y_j |\varphi_j\rangle\langle\alpha_j| \quad (10)$$

with $\{|\varphi_j\rangle\}$ being an orthonormal basis of Alice's subsystem. As a consequence, the entanglement criterion Eq. (4) turns into

$$\left| \sum_j x_j y_j \text{Tr} [|a_j\rangle\langle\alpha_j| B_1 B_2 \varrho] \right|^2 \leq \sum_j |x_j|^2 p_j \sum_k |y_k|^2 q_k \quad (11)$$

where we defined

$$\begin{aligned} p_j &\equiv \text{Tr} [|a_j\rangle\langle\alpha_j| B_2^\dagger B_2 \varrho], \\ q_k &\equiv \text{Tr} [| \alpha_k \rangle \langle \alpha_k | B_1 B_1^\dagger \varrho]. \end{aligned} \quad (12)$$

The right hand side of this equation is equal to

$$\begin{aligned} RS1 &= \sum_j |x_j|^2 |y_j|^2 p_j q_j \\ &\quad + \sum_j \sum_{k>j} |x_j|^2 |y_k|^2 p_j q_k + |x_k|^2 |y_j|^2 p_k q_j \end{aligned} \quad (13)$$

On the other hand, we can estimate the left hand side by

$$\begin{aligned} & \left| \sum_j x_j y_j \text{Tr} [|a_j\rangle\langle\alpha_j| B_1 B_2 \varrho] \right|^2 \\ & \leq \left(\sum_j |x_j| |y_j| |\text{Tr} [|a_j\rangle\langle\alpha_j| B_1 B_2 \varrho]| \right)^2 \end{aligned} \quad (14)$$

and use Eq. (4) for every single term in the summation. This leads to

$$\left| \sum_j x_j y_j \text{Tr} [|a_j\rangle\langle\alpha_j| B_1 B_2 \varrho] \right|^2 \leq \left(\sum_j |x_j| |y_j| \sqrt{p_j q_j} \right)^2 \quad (15)$$

By expanding the right hand side we get

$$\begin{aligned} RS2 &= \sum_j |x_j|^2 |y_j|^2 p_j q_j \\ &+ \sum_j \sum_{k>j} 2 |x_j x_k y_j y_k| \sqrt{p_j p_k q_j q_k} \end{aligned} \quad (16)$$

By using $r^2 + s^2 \geq 2rs$ valid for all real numbers r, s and identifying

$$r = |x_j| |y_k| \sqrt{p_j q_k} \quad (17)$$

$$s = |x_k| |y_j| \sqrt{p_k q_j} \quad (18)$$

for all arbitrary but fixed combination of j and k we find by comparing Eq. (13) with Eq. (16)

$$RS2 \leq RS1. \quad (19)$$

As a consequence, using several entanglement criteria with operators

$$A_1 = |a_j\rangle\langle\varphi_j|; \quad A_2 = |\varphi_j\rangle\langle\alpha_j| \quad (20)$$

for every j separately leads to a stronger criterion than a linear combination of these operators. \square

Naturally the same holds for Bob's subsystems which leads to the well known entanglement criterion [5, 8]

$$|\langle\alpha, \beta|\varrho|a, b\rangle_{\text{sep}}| \leq \sqrt{\langle a, \beta|\varrho|a, \beta\rangle_{\text{sep}} \langle\alpha, b|\varrho|\alpha, b\rangle_{\text{sep}}}. \quad (21)$$

Now, we turn to the question which states can be detected by Theorem II.1. Similar to the criterion from Hillery and Zubairy [10], our criterion is strongly connected to the positive partial transpose (PPT) criterion in the bipartite case [1, 2].

Theorem II.3 *The criterion Theorem II.1 for detecting bipartite entanglement detects only states with a negative partial transpose (NPT).*

Proof By using the partial transpose with respect to system A (written as T_A) acting on the state ϱ as well as

on the operators A_1 and A_2 we rewrite the expectation value

$$|\langle A_1 A_2 B_1 B_2 \rangle| = |\text{Tr} [A_2^T A_1^T B_1 B_2 \varrho^{T_A}]|. \quad (22)$$

If ϱ is PPT, then ϱ^{T_A} is a valid density operator and the Cauchy-Schwarz inequality for expectation values

$$|\text{Tr} [OP\varrho]| \leq \sqrt{\text{Tr} [OO^\dagger\varrho] \text{Tr} [P^\dagger P\varrho]} \quad (23)$$

can be applied. This leads to

$$\begin{aligned} |\text{Tr} [A_2^T A_1^T B_1 B_2 \varrho_{\text{PPT}}^{T_A}]| &\leq \sqrt{\text{Tr} [A_2^T (A_2^T)^\dagger B_1 B_1^\dagger \varrho_{\text{PPT}}^{T_A}] \\ &\times \sqrt{\text{Tr} [(A_1^T)^\dagger A_1^T B_2^\dagger B_2 \varrho_{\text{PPT}}^{T_A}]} \end{aligned} \quad (24)$$

were we identified $O = A_2^T B_1$ and $P = A_1^T B_2$. Writing the expectation value again with respect to ϱ we finally arrive at

$$|\langle A_1 A_2 B_1 B_2 \rangle_{\text{PPT}}| \leq \sqrt{\langle A_2^\dagger A_2 B_1 B_1^\dagger \rangle_{\text{PPT}} \langle A_1^\dagger A_1 B_2^\dagger B_2 \rangle_{\text{PPT}}} \quad (25)$$

which is equal to Eq. (4). As a consequence, PPT states are not able to violate Eq. (4). \square

We note that Theorem II.3 is only valid for the bipartite case. In the multipartite case, also PPT-states can be detected, since our criteria not only checks bipartite entanglement but real multipartite entanglement as we will show in the next section. However, we will once again stay in the bipartite case and identify now a few classes of entangled states which can be detected by Eq. (4). Furthermore, we will also show how to find the right operators A_j and B_j for these cases. To achieve this task, the following lemma will be helpful:

Lemma II.4 *Let $|\lambda_j^+\rangle$ be the eigenstates of ϱ^{T_A} corresponding to the positive eigenvalues and $|\lambda_j^-\rangle$ the ones corresponding to negative eigenvalues. Furthermore, we assume that there exist states $\{|a_k\rangle\}$ and $\{|b_k\rangle\}$ of Alice's and Bob's subsystems, respectively, with $k \in \{1, 2\}$ such that*

$$\langle a_2, b_1 | \lambda_j^+ \rangle = c^+ \langle a_1, b_2 | \lambda_j^+ \rangle \quad (26)$$

$$\langle a_2, b_1 | \lambda_j^- \rangle = c^- \langle a_1, b_2 | \lambda_j^- \rangle \quad (27)$$

with c^\pm independent of j and $c^+ c^- < 0$. By choosing the operators

$$A_1 = |a_1^*\rangle\langle\alpha| \quad B_1 = |b_1\rangle\langle\beta| \quad (28)$$

$$A_2 = |\alpha\rangle\langle a_2^*| \quad B_2 = |\beta\rangle\langle b_2| \quad (29)$$

with $|\alpha\rangle, |\beta\rangle$ being arbitrary states of Alice's and Bob's subsystems, respectively, Eq. (4) is violated by the state ϱ .

Proof Although the state under consideration is not PPT, we can use some calculations from the proof of Theorem II.3. By defining $p_j = \langle a_1, b_2 | \lambda_j^+ \rangle$ and $q_j = \langle a_1, b_2 | \lambda_j^- \rangle$ we obtain for the left hand side of Eq. (24)

$$LS \equiv |\langle a_1, b_2 | \varrho^{TA} | a_2, b_1 \rangle| \quad (30)$$

$$= |c^+| \sum_j \lambda_j^+ |p_j|^2 + |c^-| \sum_j |\lambda_j^-| |q_j|^2. \quad (31)$$

The right hand side becomes

$$RS \equiv \sqrt{\langle a_2, b_1 | \varrho | a_2, b_1 \rangle \langle a_1, b_2 | \varrho | a_1, b_2 \rangle} \quad (32)$$

$$= \sqrt{|c^+|^2 \sum_j \lambda_j^+ |p_j|^2 - |c^-|^2 \sum_j |\lambda_j^-| |q_j|^2} \\ \times \sqrt{\sum_j \lambda_j^+ |p_j|^2 - \sum_j |\lambda_j^-| |q_j|^2}. \quad (33)$$

By expanding the product we arrive at

$$RS = \left[|c^+|^2 \left(\sum_j \lambda_j^+ |p_j|^2 \right)^2 + |c^-|^2 \left(\sum_j |\lambda_j^-| |q_j|^2 \right)^2 \right. \\ \left. - (|c^+|^2 + |c^-|^2) \left(\sum_j \lambda_j^+ |p_j|^2 \right) \left(\sum_j |\lambda_j^-| |q_j|^2 \right) \right]^{1/2}. \quad (34)$$

With the help of $-(|c^+|^2 + |c^-|^2) \leq -2|c^+||c^-|$ we are able to estimate the right side by

$$RS \leq \left| |c^+| \sum_j \lambda_j |p_j|^2 - |c^-| \sum_j |\lambda_j^-| |q_j|^2 \right|. \quad (35)$$

A comparison of Eq. (31) and Eq. (35) shows that the left hand side of Eq. (24) is the summation of two positive numbers whereas the right side is smaller or equal than the absolute value of the difference of the same two positive numbers. As a consequence, we have $LS > RS$ and a violation of Eq. (24) which implies a violation of criterion Theorem II.1. \square

Now, we are able to identify certain classes of entangled states that can be detected with our criterion. One of these classes are bipartite qubits states:

Theorem II.5 *Every entangled two-qubit state ϱ can be detected with Theorem II.1*

Proof Every entangled two-qubit state is an NPT-state. All eigenstates $|\lambda_j^- \rangle$ corresponding to a negative eigenvalue λ_j^- of the partial transpose of ϱ form an entangled subspace [11, 12]. That means it is not possible to construct a product state by a superposition of $|\lambda_j^- \rangle$. As a consequence, for two-qubit states only one single negative eigenvalue can exist [13]. Its eigenstate is given in its Schmidt basis by

$$|\lambda^- \rangle \equiv s_0 |00\rangle + s_1 |11\rangle. \quad (36)$$

with $s_j > 0$. Since a two-qubit state is separable iff the determinant of its partial transpose is nonnegative [14], ϱ_A^T has three strictly positive eigenvalues $\lambda_j^+ > 0$. Their corresponding eigenstates are given by

$$|\lambda_k^+ \rangle \equiv r_k (s_1 |00\rangle - s_0 |11\rangle) + \gamma_k |01\rangle + \delta_k |10\rangle \quad (37)$$

with arbitrary coefficients r_k , γ_k and δ_k and at least one k for which $r_k \neq 0$. By choosing the states $|a_2, b_1\rangle = |00\rangle$ and $|a_2, b_1\rangle = |11\rangle$ we find

$$\langle 00 | \lambda_k^+ \rangle = r_k s_1, \langle 11 | \lambda_k^+ \rangle = -r_k s_0. \quad (38)$$

Therefore the constant c^+ defined in Lemma II.4 does exist for these states and is given by $c^+ = -s_1/s_0$. Similar we find $c^- = s_0/s_1$ and as a consequence Lemma II.4 can be applied. \square

We want to stress out the fact that ϱ is entangled iff the determinant of ϱ_A^T is negative is not necessary for our proof. Moreover, the existence of at least one $r_k \neq 0$ can be proven from our previous considerations:

Assume that there were an NPT two-qubit state with a vanishing eigenvalue, such that $r_k = 0$ for all k . Since the trace is preserved under partial transposition, there must exist at least one positive eigenvalue and therefore there is some $\gamma_k \neq 0$ or $\delta_k \neq 0$. Let us assume $\gamma_{k_0} \neq 0$. It follows that we can choose the product states $|a_2, b_1\rangle = |00\rangle + |01\rangle$ and $|a_1, b_2\rangle = |00\rangle - |01\rangle$ and find

$$\langle a_2 b_1 | \lambda_k^+ \rangle = \gamma_k, \langle a_1, b_2 | \lambda_k^+ \rangle = -\gamma_k. \quad (39)$$

Therefore, the constant c^+ is given by $c^+ = -1$ and similar $c^- = 1$. By using Lemma II.4 we get a violation of Eq. (24). However, in this case we would have $A_1 A_1^\dagger = |0\rangle\langle 0| = A_2^\dagger A_2$ and for these operators Eq. (4) can never be violated. As a consequence if $|\lambda_j^- \rangle \equiv \alpha |00\rangle + \beta |11\rangle$ is an eigenvector of ϱ^{TA} with a negative eigenvalue then there must be an eigenvector $|\lambda_j^+ \rangle$ with

$$\langle \lambda_j^+ | (s_1 |00\rangle - s_0 |11\rangle) \rangle \neq 0. \quad (40)$$

Also in higher dimensions there exist classes of entangled states which can be detected with our criterion. One of these classes are entangled states mixed with noise:

Theorem II.6 *Every NPT-state of the form*

$$\varrho_{wn} = p |\psi_{ent}\rangle\langle\psi_{ent}| + \frac{1-p}{D} \mathbb{1}_D, \quad (41)$$

with $\mathbb{1}_D$ being the identity matrix of dimension D of ϱ and $|\psi_{ent}\rangle$ being a pure entangled state, can be detected by Eq. (4).

To prove this theorem, we use the following fact, that can be proved by direct calculation:

Lemma II.7 *The partially transposed state ϱ_{wn}^{TA} with $|\psi_{ent}\rangle$ given in its Schmidt basis by $|\psi_{ent}\rangle = \sum_j s_j |jj\rangle$ has the following eigenvalues and eigenstates:*

$$|\lambda_{j,j}\rangle = |j,j\rangle, \quad \lambda_{j,j} = \frac{1-p}{D} + ps_j^2 \quad (42)$$

$$|\lambda_{j,k}\rangle = \frac{|j,k\rangle \pm |k,j\rangle}{\sqrt{2}}, \quad \lambda_{j,k} = \frac{1-p}{D} \pm ps_j s_k \quad (43)$$

Now we are able to prove the theorem:

Proof From the previous Lemma one finds that an eigenstate corresponding to negative eigenvalues is of the form

$$|\lambda_-\rangle = \frac{|j_0, k_0\rangle - |k_0, j_0\rangle}{\sqrt{2}} \quad (44)$$

and that

$$|\lambda_+\rangle = \frac{|j_0, k_0\rangle + |k_0, j_0\rangle}{\sqrt{2}} \quad (45)$$

is an eigenstate corresponding to a positive eigenvalue. Furthermore, $\langle j_0, k_0 | \lambda_n \rangle = \langle k_0, j_0 | \lambda_n \rangle = 0$ for all other eigenstates $|\lambda_n\rangle$. By choosing $|a_2, b_1\rangle = |j_0, k_0\rangle$ and $|a_1, b_2\rangle = |k_0, j_0\rangle$ we find

$$\langle a_2, b_1 | \lambda_- \rangle = -\langle a_1, b_2 | \lambda_- \rangle \quad (46)$$

$$\langle a_2, b_1 | \lambda_+ \rangle = +\langle a_1, b_2 | \lambda_+ \rangle \quad (47)$$

$$\langle a_2, b_1 | \lambda_n \rangle = \pm \langle a_1, b_2 | \lambda_n \rangle = 0. \quad (48)$$

As a consequence the constants c^+ and c^- of Lemma II.4 do exist and therefore the entanglement of ϱ_{wn}^{PT} can be detected with the help of Theorem II.1. \square

It is important to point out a consequence of the optimality of choosing $A_{1/2}$ and $B_{1/2}$ such that the criterion takes the form of eq. (21). There are only up to four independent vectors (in the previous notation $|a\rangle, |b\rangle, |\alpha\rangle, |\beta\rangle$) appearing in the inequality. If we denote $|\alpha\rangle = \lambda|a\rangle + \sqrt{1-|\lambda|^2}|a^\perp\rangle$ with $\lambda = \langle a|\alpha\rangle$ it becomes obvious that the criterion is invariant under prior projection into the qubit subspace on Alice's side given by $\mathbb{1}_2 = |a\rangle\langle a| + |a^\perp\rangle\langle a^\perp|$. As the same holds for Bob's side we can conclude that the bipartite version of this theorem is actually equivalent to detecting entanglement in a 2×2 dimensional subspace. From Ref. [15] it follows that any violation of the criterion implies one-copy distillability, which on the other hand implies the impossibility of detecting some NPT states, such as the well known Werner state [16]. In fact it also implies that we can strictly improve the detection strength of the criterion by applying it on multiple copies of the state, as there are known examples of states that are two-, but not one-copy distillable. However some NPT states are most likely beyond the reach of our criterion as e.g. the Werner state is conjectured to be NPT bound entangled, i.e. even up to infinitely many copies do not have an entangled 2×2 subspace for a specific region of parameters. A natural extension of our theorem that would close this

(small) gap for bipartite systems is an extension to $D \times D$ -dimensional subspaces, which could be written as

$$\text{Det} \begin{pmatrix} \rho_{i_1 i_1 i_1 i_1} & \rho_{i_1 i_2 i_2 i_1} & \cdots & \rho_{i_1 i_D i_D i_1} \\ \rho_{i_2 i_1 i_1 i_2} & \rho_{i_2 i_2 i_2 i_2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{i_D i_1 i_1 i_D} & \cdots & \cdots & \rho_{i_D i_D i_D i_D} \end{pmatrix} \geq 0, \quad (49)$$

with the notation $\langle ab | \rho_{PPT} | \alpha\beta \rangle = \rho_{ab\alpha\beta}$. This criterion is a natural extension of our main theorem to higher dimensions and in this form is in principle capable of detecting all NPT states, not only one-copy distillable ones. The caveat here is however, that due to the non-convex structure we can neither make use of the Cauchy-Schwarz inequality, nor find an analogue extension to the more interesting case of many particles. So instead we focus on our main theorem, which allows for a straightforward generalization to multipartite systems.

B. Multipartite entanglement

Pure product states and mixed product states of the form $\varrho = \bigotimes_{j=1}^N \varrho_j$ with ϱ_j do not contain any correlations (nc). The generalization of Eq. (5) to multipartite systems for these states is straightforward:

$$|\langle \bigotimes_{k=1}^N P_k Q_k \rangle_{nc}| \leq \prod_{k=1}^N \sqrt{\langle P_k P_k^\dagger \rangle_{nc} \langle Q_k^\dagger Q_k \rangle_{nc}} \quad (50)$$

with operators P_k and Q_k acting on subsystem k . Again, we can combine expectation values of different subsystems and arrive at a generalization of Theorem II.1:

$$|\langle \bigotimes_{k=1}^N P_k Q_k \rangle_{nc}| \leq \prod_{k=1}^N \sqrt{\langle P_k P_k^\dagger Q_{k+1}^\dagger Q_{k+1} \rangle_{nc}}. \quad (51)$$

where we defined $Q_{N+1} = Q_1$. Also other combination of expectation values are possible. However, these inequalities are not convex and therefore in general not valid for mixed separable states.

To generalize entanglement criteria of the form of Eq. (51) to mixed fully separable states, we have to use the Hölder inequality. We will explain the generalization using the example of Eq. (51) for a tripartite state with the operators A_j , B_j and C_j acting on Alice's, Bob's and Charlie's subsystem, respectively.

1. Scheme to develop entanglement criteria

Performing the following steps will result in an inequality to detect entanglement. We assume that we start with an fully separable state

$$\varrho = \sum_j p_j |\psi_j^{\text{ps}}\rangle\langle\psi_j^{\text{ps}}| \quad (52)$$

where the $|\psi_j^{\text{ps}}\rangle$ are pure multiparticle product states as above.

- Write the expectation value $\langle A_1 A_2 B_1 B_2 C_1 C_2 \rangle$ of the fully separable state as a convex combination of expectation values for pure product states $|\psi_j^{\text{ps}}\rangle$.

$$\langle A_1 A_2 B_1 B_2 C_1 C_2 \rangle_{\text{sep}} = \sum_j p_j \langle A_1 A_2 B_1 B_2 C_1 C_2 \rangle_{\text{ps},j} \quad (53)$$

with $\sum_j p_j = 1$ and $\langle \dots \rangle_{\text{ps},j}$ denoting the expectation value of state $|\psi_j^{\text{ps}}\rangle$.

- Write every expectation value $\langle \dots \rangle_{\text{ps},j}$ as expectation values of single subsystems and apply the Cauchy Schwarz inequality to each of them

$$|\langle A_1 A_2 B_1 B_2 C_1 C_2 \rangle_{\text{sep}}| \leq \sum_j p_j \sqrt{\langle A_1 A_1^\dagger \rangle_j \langle A_2 A_2^\dagger \rangle_j \dots} \quad (54)$$

Since operators of the form $O = DD^\dagger$ are positive operators, it is also possible to increase the number of expectation values by using $\langle O \rangle = \sqrt[n]{\langle O \rangle^n}$.

- Combine arbitrary expectation values of different subsystems, for example

$$|\langle A_1 A_2 B_1 B_2 C_1 C_2 \rangle_{\text{sep}}| \leq \sum_j p_j \sqrt{\langle A_1 A_1^\dagger B_2^\dagger B_2 \rangle_j \langle B_1 B_1^\dagger C_2^\dagger C_2 \rangle_j \langle C_1 C_1^\dagger A_2^\dagger A_2 \rangle_j} \quad (55)$$

It is also possible to combine more than two subsystems in a single expectation value.

- Use the generalized Hölder inequality [17]

$$\sum_j p_j x_j y_j \leq \left(\sum_j p_j x_j^{1/r} \right)^r \left(\sum_j p_j y_j^{1/s} \right)^s \quad (56)$$

with $r + s = 1$ and $p_j \geq 0$ to separate the summation of each expectation value. Here, the Hölder inequality may be used several times. In our example, two applications of the inequality lead to:

$$\begin{aligned} |\langle A_1 A_2 B_1 B_2 C_1 C_2 \rangle_{\text{sep}}| &\leq \left(\sum_j p_j \langle A_1 A_1^\dagger B_2^\dagger B_2 \rangle_{\text{ps},j}^{3/2} \right)^{1/3} \\ &\quad \times \left(\sum_j p_j \langle B_1 B_1^\dagger C_2^\dagger C_2 \rangle_{\text{ps},j}^{3/2} \right)^{1/3} \\ &\quad \times \left(\sum_j p_j \langle C_1 C_1^\dagger A_2^\dagger A_2 \rangle_{\text{ps},j}^{3/2} \right)^{1/3}. \end{aligned} \quad (57)$$

- Use the inequality

$$\langle O \rangle^x \leq \langle O^x \rangle \quad (58)$$

for positive operators O and $x \geq 1$ to make the expectation values appear only linearly. Then, everything can be written in terms of ϱ again, and the p_j disappear. In our example we finally arrive at

$$\begin{aligned} &|\langle A_1 A_2 B_1 B_2 C_1 C_2 \rangle_{\text{sep}}|^2 \\ &\leq \sqrt[3]{\langle (A_1 A_1^\dagger B_2^\dagger B_2)^{3/2} \rangle_{\text{sep}} \langle (B_1 B_1^\dagger C_2^\dagger C_2)^{3/2} \rangle_{\text{sep}}} \\ &\quad \times \sqrt[3]{\langle (C_1 C_1^\dagger A_2^\dagger A_2)^{3/2} \rangle_{\text{sep}}}, \end{aligned} \quad (59)$$

which is valid for all separable mixed states but can be violated by entangled states.

2. Application

In the same way, the criterion

$$\begin{aligned} &|\langle A_1 A_2 B_1 B_2 C_1 C_2 \rangle_{\text{sep}}| \\ &\leq \sqrt[4]{\langle A_1 A_1^\dagger B_1 B_1^\dagger C_2^\dagger C_2 \rangle_{\text{sep}} \langle A_1 A_1^\dagger B_2^\dagger B_2 C_1 C_1^\dagger \rangle_{\text{sep}}} \\ &\quad \times \sqrt[4]{\langle A_2^\dagger A_2 B_1 B_1^\dagger C_1 C_1^\dagger \rangle_{\text{sep}} \langle A_2^\dagger A_2 B_2^\dagger B_2 C_2^\dagger C_2 \rangle_{\text{sep}}} \end{aligned} \quad (60)$$

can be shown. This criterion checks all possible bipartitions simultaneously and cannot be seen as a combination of criteria for bipartite entanglement. Therefore it is possible to detect entanglement of states, which are biseparable under every bipartition but not fully separable, for example the bound entangled states from Ref. [18]

$$\varrho_{abc} = \frac{1}{n} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/a & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (61)$$

for $abc \neq 1$ with $n = 2 + a + b + c + 1/a + 1/b + 1/c$. These states are known to be separable for any bipartition, but not fully separable. By choosing

$$\begin{aligned} A_1 &= B_1 = C_1 = |1\rangle\langle 0| \\ A_2 &= B_2 = C_2 = |0\rangle\langle 0|. \end{aligned} \quad (62)$$

we detect entanglement if $1 < (abc)^{-1/4}$. A second criterion can be gained by combining the expectation values of single systems in another way. This second criterion detect entanglement if $1 < (abc)^{+1/4}$. As a consequence, with the help of the CS inequality it can be proven that the state ϱ_{abc} is entangled for $abc \neq 1$. A similar result was obtained in Ref.[5]. However, as we will show in the next section, our criterion is in general stronger.

Since the operator $|1\rangle\langle 0|$ is a non-hermitian, its expectation value cannot be directly measured. However, since $|1\rangle\langle 0| = \sigma_x - i\sigma_y$ the expectation values can be

determined by measuring the expectation values of the Pauli matrices in x and y direction.

A second interesting example is the bound entangled state from Ref. [19],

$$\varrho_\alpha = \frac{1}{8+8\alpha} \begin{pmatrix} 4+\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & \alpha & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & \alpha & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 4+\alpha \end{pmatrix} \quad (63)$$

This is a valid state for $2 \leq \alpha$. This state is entangled (but separable for any bipartition) for $2 \leq \alpha \leq 2\sqrt{2}$ and separable for $2\sqrt{2} \leq \alpha$ [19, 20]

We use our criterion Eq. (60) with operators similar to Theorem II.2. Therefore, we define the quantity

$$\begin{aligned} E \equiv & |\langle \alpha_1, \beta_1, \gamma_1 | \varrho | \alpha_2, \beta_2, \gamma_2 \rangle| \\ & - \left(\sqrt[4]{\langle \alpha_1, \beta_1, \gamma_2 | \varrho | \alpha_1, \beta_1, \gamma_2 \rangle \langle \alpha_1, \beta_2, \gamma_1 | \varrho | \alpha_1, \beta_2, \gamma_1 \rangle} \right. \\ & \times \left. \sqrt[4]{\langle \alpha_2, \beta_1, \gamma_1 | \varrho | \alpha_2, \beta_1, \gamma_1 \rangle \langle \alpha_2, \beta_2, \gamma_2 | \varrho | \alpha_2, \beta_2, \gamma_2 \rangle} \right) \end{aligned} \quad (64)$$

which indicates entanglement if $E > 0$. Numerical maximization of E leads to the optimal measurement basis

$$\begin{aligned} |\alpha_1\rangle &= (|0\rangle - e^{-i\varphi}|1\rangle)/\sqrt{2} \\ |\alpha_2\rangle &= (|0\rangle + e^{-i(\frac{\pi}{2}-\varphi)}|1\rangle)/\sqrt{2} \\ |\beta_1\rangle &= (|0\rangle - e^{+i(\frac{\pi}{2}-\varphi)}|1\rangle)/\sqrt{2} \\ |\beta_2\rangle &= (|0\rangle + e^{+i\varphi}|1\rangle)/\sqrt{2} \\ |\gamma_1\rangle &= (|0\rangle + e^{-i\varphi}|1\rangle)/\sqrt{2} \\ |\gamma_2\rangle &= (|0\rangle - e^{-i(\frac{\pi}{2}-\varphi)}|1\rangle)/\sqrt{2} \end{aligned} \quad (65)$$

with the phase $\varphi \approx 0.138\pi$ for $\alpha = 2$. Here, the two measurement directions in a single subsystems are not orthogonal anymore. The result of the entanglement detection is shown in Fig. 1. For $\alpha < 2.4$ our criterion detects entanglement independently of whether we use the measurement basis optimized for $\alpha = 2$ (+) or optimized the measurement basis for each α separately (x). For $\alpha < 2.4$ both methods lead to the same quantity E and to the same measurement basis. For $2.4 < \alpha$ also the optimized measurement basis does not detect the states, since it leads to $E = 0$. Note that in contrast to ϱ_{abc} , the entanglement of this state cannot be detected by the criteria developed from Ref.[5], but there exist refined criteria which detect the entanglement in the whole region $2 \leq \alpha \leq 2\sqrt{2}$ [20].

III. CONNECTIONS TO EXISTING CRITERIA

As explained in the introduction, one of the main motivations of our paper is to present an unified view on

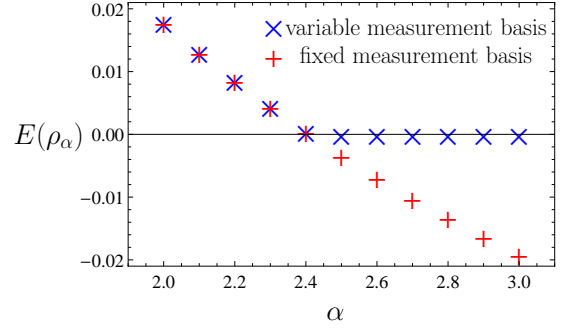


FIG. 1: Entanglement detection of ϱ_α Eq. (63) with the help of Eq. (60). $E > 0$ indicates entanglement. For \times the measurement basis was optimized for each α , whereas for $+$ we used the optimized basis for $\alpha = 2$ for all α . See the text for further details.

several existing entanglement criteria. So in this section we show that several other entanglement criteria based on inequalities are applications of Eq. (4), although they were originally proven in a different way.

A. The criterion of Hillery and Zubairy

The entanglement criterion

$$|\langle A^\dagger B \rangle_{\text{sep}}|^2 \leq \langle A^\dagger A B^\dagger B \rangle_{\text{sep}} \quad (66)$$

from Ref. [10] is a special case of Eq. (4) where we have set A_2 and B_1 equal to unity and $A_1 = A^\dagger$ and $B_2 = B$. By identifying $A = a^m$ with a being the annihilation operator of system A and $B = (b^\dagger)^n$ with b^\dagger being the creation operator of system B we rederive their original criterion

$$|\langle a^m (b^\dagger)^n \rangle_{\text{sep}}|^2 \leq \langle (a^\dagger)^m a^m (b^\dagger)^n b^n \rangle_{\text{sep}} \quad (67)$$

given in Ref. [4]. On the other hand,

$$|\langle a^m b^n \rangle_{\text{sep}}|^2 \leq \langle (a^\dagger)^m a^m \rangle_{\text{sep}} \langle (b^\dagger)^n b^n \rangle_{\text{sep}} \quad (68)$$

which was also derived in Ref. [4] belongs to the special case

$$|\langle AB \rangle_{\text{sep}}|^2 \leq \langle A^\dagger A \rangle_{\text{sep}} \langle B^\dagger B \rangle_{\text{sep}} \quad (69)$$

where we have set A_1 and B_1 equal to unity.

Let us compare our entanglement criterion Eq. (4) with the criteria of the type Eq. (69) and Eq. (66) derived by Hillery and Zubairy for two qubit systems. By choosing

$$A_1 = |1\rangle\langle 0|, A_2 = |0\rangle\langle 0|, B_1 = |1\rangle\langle 0|, B_2 = |0\rangle\langle 0| \quad (70)$$

we obtain from Eq. (4) that all separable states ϱ obey

$$|\varrho_{00,11}^{\text{sep}}|^2 \leq \varrho_{01,01}^{\text{sep}} \varrho_{10,10}^{\text{sep}}. \quad (71)$$

On the other hand Eq. (66) and Eq. (69) transform for the choice of $A = A_1$ and $B = B_1$ into

$$|\varrho_{00,11}^{\text{sep}}|^2 \leq \varrho_{01,01}^{\text{sep}} \quad (72)$$

$$|\varrho_{00,11}^{\text{sep}}|^2 \leq (\varrho_{10,10}^{\text{sep}} + \varrho_{11,11}^{\text{sep}})(\varrho_{01,01}^{\text{sep}} + \varrho_{11,11}^{\text{sep}}). \quad (73)$$

Due to the normalization of the state the coefficients of the matrix obey $0 \leq \varrho_{jk,jk} \leq 1$ and therefore these criteria are weaker than Eq. (4). However, Eq. (66) requires only the estimation of a single expectation value or matrix entry instead of two for our criterion, and Eq. (69) only requires expectation values depending on single subsystems rather than correlations of both systems required for Eq. (4).

As suggested in Ref. [4] Eq. (66) can be generalized to

$$|\langle \prod_{k=1}^N A_k \rangle_{\text{sep}}|^2 \leq \langle \prod_{k=1}^j A_k^\dagger A_k \prod_{k=j+1}^N A_k A_k^\dagger \rangle_{\text{sep}} \quad (74)$$

with A_k being an operator acting on system k . This inequality holds true not only for separable states, but also for biseparable states with respect to the partition $1, 2, \dots, j | j+1, \dots, N$. As a consequence, the inequality checks only if the state is biseparable with respect to a certain bipartition and does not check for real multipartite entanglement.

B. The criterion of Hillery et al.

In Ref. [6] the authors used a similar method to ours to generalize Eq. (69) to multipartite states: (i) First, they write the separable state as a convex set of pure product states. (ii) Then, they write every expectation value as the product of expectation values of single subsystems and apply the CS inequality to each of them. (iii) Finally, they use the Hölder inequality Eq. (56) to imply convexity. In this way they derive the criterion

$$|\langle \prod_{k=1}^n A_k \rangle_{\text{sep}}| \leq \prod_{k=1}^n \langle (A_k^\dagger A_k)^{n/2} \rangle_{\text{sep}}^{1/n} \quad (75)$$

where A_k denotes an operator acting on subsystem k . For their second criterion

$$|\langle \prod_{k=1}^n A_k \rangle_{\text{sep}}| \leq \langle \left(\frac{1}{n} \sum_{k=1}^n A_k^\dagger A_k \right)^{n/2} \rangle_{\text{sep}} \quad (76)$$

they used in addition that the geometric mean is smaller or equal than the arithmetic mean between step (ii) and (iii). For some states, both criteria are equal, for some Eq. (75) is stronger and for others Eq. (76) is stronger. However, both criteria used again only a single operator per subsystem. Furthermore, after step (ii) the property of product states was not used anymore whereas in our criteria we used these properties again to recombine expectation values. Therefore, our criterion is able to detect PPT states like ϱ_{abc} defined in Eq. (61) whereas Eq. (75) cannot, which follows from the following theorem:

Theorem III.1 *Assume a state ϱ which is biseparable under some partitions in the following way: For any pair of subsystems k and j there exists a bipartition $M|\bar{M}$ of the N particles such that $k \in M$ and $j \in \bar{M}$ and the*

state ϱ is biseparable for this bipartition $M|\bar{M}$. Then the criterion Eq. (75) cannot detect the entanglement of this state.

Note that by far not all bipartitions have to be separable to fulfill this condition, the minimal number of different bipartitions needed for this assumption scales like $\log_2(N)$.

Proof Without loss of generality we assume there exists a bipartition $A_1, \dots, A_l | A_{l+1}, \dots, A_n$, and therefore

$$|\langle \prod_{k=1}^n A_k \rangle| \leq \sum_j p_j \langle \prod_{k=1}^l A_k^\dagger A_k \rangle_j^{1/2} \langle \prod_{k=l+1}^n A_k^\dagger A_k \rangle_j^{1/2}. \quad (77)$$

With the help of the generalized Hölder inequality Eq. (56) we obtain

$$\begin{aligned} |\langle \prod_{k=1}^n A_k \rangle| &\leq \left(\sum_j p_j \langle \prod_{k=1}^l A_k^\dagger A_k \rangle_j^{n/2l} \right)^{l/n} \\ &\quad \times \left(\sum_j p_j \langle \prod_{k=l+1}^n A_k^\dagger A_k \rangle_j^{n/2(n-l)} \right)^{(n-l)/n}. \end{aligned} \quad (78)$$

by choosing $r = l/n$ and $s = (n-l)/n$. Again, with the help of Eq. (58) we find

$$\begin{aligned} |\langle \prod_{k=1}^n A_k \rangle| &\leq \left(\langle \prod_{k=1}^l (A_k^\dagger A_k)^{n/2l} \rangle \right)^{l/n} \\ &\quad \times \left(\langle \prod_{k=l+1}^n (A_k^\dagger A_k)^{n/2(n-l)} \rangle \right)^{(n-l)/n}. \end{aligned} \quad (79)$$

For each of the expectation values $\langle \prod A_k^\dagger A_k \rangle$ exist again a bipartition (with different p_j and ψ_j) and therefore we arrive at

$$\begin{aligned} &\left(\langle \prod_{k=1}^l (A_k^\dagger A_k)^{n/2l} \rangle \right)^{l/n} \\ &= \left(\sum_j p_j \langle \prod_{k=1}^x (A_k^\dagger A_k)^{n/2l} \rangle_j \langle \prod_{k=x+1}^l (A_k^\dagger A_k)^{n/2l} \rangle_j \right)^{l/n} \\ &\leq \left(\langle \prod_{k=1}^x (A_k^\dagger A_k)^{n/2x} \rangle \right)^{x/n} \\ &\quad \times \left(\langle \prod_{k=x+1}^l (A_k^\dagger A_k)^{n/2(l-x)} \rangle \right)^{(l-x)/n} \end{aligned} \quad (80)$$

by using again the generalized Hölder equation, now with $r = x/l$ and $s = (l-x)/l$ and Eq. (58). This argument can be applied until all expectation values are expectation values of a single subsystem and as a consequence we derive

$$|\langle \prod_{k=1}^n A_k \rangle| \leq \prod_{k=1}^n \langle (A_k^\dagger A_k)^{n/2} \rangle^{1/n}. \quad (81)$$

This shows that the entanglement of an N -partite state ϱ cannot be detected by Eq. (75), if for every two subsystem i and j , there exist a bipartition of ϱ where i and j can be separated. Furthermore, it shows that the criterion Eq. (75) is just a recursive application of the criterion Eq. (69) for bipartite entanglement from Hillery and Zubairy.

C. The criteria from Shchukin and Vogel

The criterion

$$|\langle a^{\dagger m} a^n a^{\dagger p} a^q b^{\dagger s} b^r b^{\dagger k} b^l \rangle_{\text{sep}}|^2 \leq \langle a^{\dagger m} a^n a^{\dagger n} a^m b^{\dagger l} b^k b^{\dagger k} b^l \rangle_{\text{sep}} \times \langle a^{\dagger q} a^p a^{\dagger p} a^q b^{\dagger s} b^r b^{\dagger r} b^s \rangle_{\text{sep}} \quad (82)$$

derived by Shchukin and Vogel in [3] was originally proved with the help of the PPT criterion [1, 2, 21] and the matrix of moments.

Similarly, our criterion Eq. (4) follows also directly from the PPT criterion in the bipartite case. However, our criterion does not detect all NPT states. Nevertheless, Eq. (4) can be used to prove the criteria Eq. (82) by identifying

$$A_1 = (a^{\dagger})^m a^n, \quad A_2 = (a^{\dagger})^p a^q, \quad (83)$$

$$B_1 = (b^{\dagger})^s b^r, \quad B_2 = (b^{\dagger})^k b^l. \quad (84)$$

As a consequence, all entangled states which can be detected with the criterion Eq. (82) derived by Shchukin and Vogel can be also detected with our criterion Eq. (4).

D. The criteria by Gühne and Seevinck

In Ref. [5] the investigation of multiparticle entanglement was started by deriving criteria for bipartite entanglement,

$$|\varrho_{1,8}^{\text{sep}}| \leq \sqrt{\varrho_{2,2}^{\text{sep}} \varrho_{7,7}^{\text{sep}}} \quad (85)$$

$$|\varrho_{1,8}^{\text{sep}}| \leq \sqrt{\varrho_{3,3}^{\text{sep}} \varrho_{6,6}^{\text{sep}}} \quad (86)$$

$$|\varrho_{1,8}^{\text{sep}}| \leq \sqrt{\varrho_{4,4}^{\text{sep}} \varrho_{5,5}^{\text{sep}}} \quad (87)$$

with the standard product basis $\{|000\rangle, |001\rangle, \dots, |111\rangle\}$. Although they were proven in another way, they directly follow from our criterion Eq. (4) by interpreting the systems A and B as two sets of systems instead of two single systems.

Using the fact that the geometric mean is smaller or equal to the arithmetic mean, which denotes

$$\prod_{j=1}^N a_k^{1/N} \leq \frac{1}{N} \sum_{j=1}^N a_k \quad (88)$$

these three equations lead to the criteria

$$|\varrho_{1,8}^{\text{sep}}| \leq \frac{1}{2}(\varrho_{2,2}^{\text{sep}} + \varrho_{7,7}^{\text{sep}}) \quad (89)$$

$$|\varrho_{1,8}^{\text{sep}}| \leq \frac{1}{2}(\varrho_{3,3}^{\text{sep}} + \varrho_{6,6}^{\text{sep}}) \quad (90)$$

$$|\varrho_{1,8}^{\text{sep}}| \leq \frac{1}{2}(\varrho_{4,4}^{\text{sep}} + \varrho_{5,5}^{\text{sep}}) \quad (91)$$

used also in Ref. [7].

By a convex combination of the three equations Eq. (85), Eq. (86) and Eq. (87) one can derive the condition

$$|\varrho_{1,8}^{\text{sep}}| \leq \sqrt{\varrho_{2,2}^{\text{sep}} \varrho_{7,7}^{\text{sep}}} + \sqrt{\varrho_{3,3}^{\text{sep}} \varrho_{6,6}^{\text{sep}}} + \sqrt{\varrho_{4,4}^{\text{sep}} \varrho_{5,5}^{\text{sep}}} \quad (92)$$

which is valid for all biseparable three qubit states, and is only violated by fully entangled states. This, of course, can not be derived by our framework, since we are only dealing with criteria excluding full separability.

However, by multiplying the three equations Eq. (85), Eq. (86) and Eq. (87) one finds the inequality

$$|\varrho_{1,8}^{\text{sep}}| \leq (\varrho_{2,2}^{\text{sep}} \varrho_{3,3}^{\text{sep}} \varrho_{4,4}^{\text{sep}} \varrho_{5,5}^{\text{sep}} \varrho_{6,6}^{\text{sep}} \varrho_{7,7}^{\text{sep}})^{1/6}. \quad (93)$$

which is valid for all fully separable states. This inequality can be proven also directly by using our scheme from Sec. IIB 1 to develop the criterion

$$\begin{aligned} & |\langle A_1 A_2 B_1 B_2 C_1 C_2 \rangle_{\text{sep}}| \\ & \leq \sqrt[6]{\langle A_2^{\dagger} A_2 B_2^{\dagger} B_2 C_1^{\dagger} C_1 \rangle_{\text{sep}} \langle A_2^{\dagger} A_2 B_1^{\dagger} B_1 C_2^{\dagger} C_2 \rangle_{\text{sep}}} \\ & \quad \times \sqrt[6]{\langle A_1 A_1^{\dagger} B_2^{\dagger} B_2 C_1^{\dagger} C_1 \rangle_{\text{sep}} \langle A_1 A_1^{\dagger} B_2^{\dagger} B_2 C_2^{\dagger} C_2 \rangle_{\text{sep}}} \\ & \quad \times \sqrt[6]{\langle A_2^{\dagger} A_2 B_1^{\dagger} B_1 C_1^{\dagger} C_1 \rangle_{\text{sep}} \langle A_1 A_1^{\dagger} B_1^{\dagger} B_1 C_2^{\dagger} C_2 \rangle_{\text{sep}}}, \end{aligned} \quad (94)$$

and by using the operators defined in Eq. (62) and Eq. (62). Since Eq. (93) is a product of three criteria for bi-separability, it cannot detect the entanglement of PPT states like ϱ_{abc} Eq. (61). However, as noted in Ref. [5] in Eq. (93) it is possible to substitute certain matrix entries $\varrho_{j,j}$ by others, for example $\varrho_{2,2} \varrho_{3,3} \rightarrow \varrho_{1,1} \varrho_{4,4}$ to get different criteria. The substitution of matrix entries is equivalent to a different combination of the operators A_j , B_j and C_j in our scheme. These new criteria are able to detect PPT states, for example the criterion

$$|\varrho_{1,8}^{\text{sep}}| \leq (\varrho_{1,1}^{\text{sep}} (\varrho_{4,4}^{\text{sep}})^2 \varrho_{5,5}^{\text{sep}} \varrho_{6,6}^{\text{sep}} \varrho_{7,7}^{\text{sep}})^{1/6} \quad (95)$$

derived by substituting $\varrho_{2,2} \varrho_{3,3} \rightarrow \varrho_{1,1} \varrho_{4,4}$ in Eq. (93), leads to $1 \leq (abc)^{-1/6}$ for the PPT-state ϱ_{abc} given in Eq. (61) which is violated for $abc > 1$. With a second similar equations, the entanglement can be detected for $1 \leq (abc)^{1/6}$. As a consequence, the criteria of Ref. [5] detect entanglement if $abc \neq 1$ similar to our criterion shown in Sec. IIB 2. However, by translating our criterion from Sec. IIB 2 into density matrix representation

$$|\varrho_{1,8}^{\text{sep}}| \leq \sqrt[4]{\varrho_{1,1}^{\text{sep}} \varrho_{4,4}^{\text{sep}} \varrho_{6,6}^{\text{sep}} \varrho_{7,7}^{\text{sep}}} \quad (96)$$

we see that the criterion of Ref. [5] is a weighted mean of the criteria Eq. (87) and Eq. (96). Therefore Eq. (95) is only equally or less strong than testing Eq. (87) and Eq. (96) separately. This was also noted in Ref. [20].

E. The criterion by Huber et al.

In Ref. [8] the criterion

$$\begin{aligned} & \sqrt{\text{Re} [\langle \Phi_{\text{ps}} | (\mathbb{1}_A \otimes \Pi_B)^\dagger \varrho_{\text{sep}}^{\otimes m} (\Pi_A \otimes \mathbb{1}_B) | \Phi_{\text{ps}} \rangle]} \\ & \leq \sqrt{\langle \Phi_{\text{ps}} | \varrho_{\text{sep}}^{\otimes m} | \Phi_{\text{ps}} \rangle} \end{aligned} \quad (97)$$

where $\varrho_{\text{sep}}^{\otimes m}$ is an m -fold tensor product of the density matrix ϱ_{sep} , $|\Phi_{\text{ps}}\rangle$ being a product state of the m -tupled system and Π being the cyclic permutation operator. For the case $m = 2$ this criterion transforms exactly into Eq. (21) and is therefore equivalent to our criterion if one chooses the operators to be projectors on a single state.

For the case $m > 2$ the criteria can be led back to a combination of the criterion with $m = 2$ and different states $|\Phi_{\text{ps}}\rangle$. Therefore, the criteria for $m > 2$ do not show any advances compared to the criterion for $m = 2$.

In the multipartite case, the generalization of the criterion for $m = 2$ reads

$$\begin{aligned} & \sqrt{\langle \Phi_{\text{ps}} | \varrho_{\text{bs}}^{\otimes m} \Pi | \Phi_{\text{ps}} \rangle} \\ & - \sum_j \sqrt{\langle \Phi_{\text{ps}} | \mathcal{P}_j^\dagger \varrho_{\text{bs}}^{\otimes m} \mathcal{P}_j | \Phi_{\text{ps}} \rangle} \leq 0 \end{aligned} \quad (98)$$

which is valid for all biseparable (bs) states. Here, the operator Π performs simultaneous permutations on all subsystems and \mathcal{P}_j performs only a permutation on the subsystem A_j for the different bipartitions j .

In a similar way, our criteria can be transformed to detect only genuine multipartite entanglement. Let X_k and Y_k be operators acting on subsystem k . Furthermore, let S_j be the set of all pure biseparable states which are biseparable under the bipartition $A_j|B_j$. As a consequence, every biseparable state

$$\varrho_{\text{bs}} \equiv \sum_j \sum_{s_j \in S_j} p_{s_j} |s_j\rangle \langle s_j| \quad (99)$$

can be written as a convex combination of states s_j belonging to different sets S_j with weights p_{s_j} . In this case we find the inequality

$$\begin{aligned} |\langle \prod_k X_k Y_k \rangle_{\text{bs}}| & \leq \sum_j \sum_{s_j \in S_j} p_{s_j} \sqrt{\langle \prod_{k \in A_j} X_k X_k^\dagger \prod_{k \in B_j} Y_k^\dagger Y_k \rangle_{s_j}} \\ & \times \sqrt{\langle \prod_{k \in A_j} Y_k^\dagger Y_k \prod_{k \in B_j} X_k X_k^\dagger \rangle_{s_j}}. \end{aligned} \quad (100)$$

Since $X_k X_k^\dagger$ and $Y_k^\dagger Y_k$ are positive operators we increase the right side of this inequality by increasing the summation range from S_j to $S = \cup S_j$. As a consequence, we obtain the criterion

$$\begin{aligned} |\langle \prod_k X_k Y_k \rangle_{\text{bs}}| & \leq \sum_j \sqrt{\langle \prod_{k \in A_j} X_k X_k^\dagger \prod_{k \in B_j} Y_k^\dagger Y_k \rangle_{\text{bs}}} \\ & \times \sqrt{\langle \prod_{k \in A_j} Y_k^\dagger Y_k \prod_{k \in B_j} X_k X_k^\dagger \rangle_{\text{bs}}}. \end{aligned} \quad (101)$$

for genuine multipartite entanglement.

IV. CONCLUSION

In this paper we have shown how to develop entanglement criteria with the help of the Cauchy-Schwarz and the Hölder inequality and the properties of separable states. In the bipartite case, our criterion is strongly connected to the PPT-criterion and only NPT-states can be detected. We demonstrate that all two-qubit states and all NPT-entangled states mixed with white noise can be detected with our criteria. However, not all NPT-states can be detected with our criterion, since it operates in a 2×2 dimensional subspace only. However, we suggested some ideas for an additional entanglement criterion which may detect such states.

We generalized our criteria to the multipartite case with the help of the Hölder inequality. We demonstrated that with our scheme it is possible to detect the entanglement of states which are separable under every bipartite split. As a consequence, our criteria are not restricted to NPT entangled states in the multipartite case. We also explained how to transform our criteria in such a way that they detect only genuine multipartite entanglement.

Furthermore, we showed that some already existing criteria for bipartite and multipartite entanglement which have been proven already with other methods, are direct consequences of the Cauchy-Schwarz inequality. As a consequence, our method to derive entanglement criteria is more fundamental. We demonstrated that criteria which use two operators per subsystems are in general stronger than criteria which use only a single operator per subsystem. As a consequence, our methods lead to stronger entanglement criteria.

V. ACKNOWLEDGEMENT

S.W. thanks Suhail Zubairy for fruitful discussions. This work has been supported by the EU (Marie Curie CIG 293993/ENFOQI and Marie Curie IEF 302021/QUACOCOS), the BMBF (Chist-Era Project QUASAR), the FQXi Fund (Silicon Valley Community Foundation), and the DFG.

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- [1] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
 - [2] P. Horodecki, R. Horodecki, and M. Horodecki, Phys. Lett. A **223**, 1 (1996).
 - [3] E. Shchukin and W. Vogel, Phys. Rev. Lett. **95**, 230502 (2005).
 - [4] M. Hillery and M. S. Zubairy, Phys. Rev. Lett. **96**, 050503 (2006).
 - [5] O. Gühne and M. Seevinck, New J. of Phys. **12**, 053002 (2010).
 - [6] M. Hillery, H. T. Dung, and H. Zheng, Phys. Rev. A **81**, 062322 (2010).
 - [7] W. Dür and J. I. Cirac, Phys. Rev. A **61**, 042314 (2000).
 - [8] M. Huber, F. Mintert, A. Gabriel, and B. C. Hiesmayr, Phys. Rev. Lett. **104**, 210501 (2010).
 - [9] A.-L. Cauchy, *Analyse algébrique* (Imprimerie Royale, Paris, 1821).
 - [10] M. Hillery, H. T. Dung, and J. Niset, Phys. Rev. A **80**, 052335 (2009).
 - [11] N. Johnston, Phys. Rev. A **87**, 064302 (2013).
 - [12] S. Rana, Phys. Rev. A **87**, 054301 (2013).
 - [13] A. Sanpera, R. Tarrach, and G. Vidal, Phys. Rev. A **58**, 826 (1998).
 - [14] R. Augusiak, M. Demianowicz, and P. Horodecki, Phys. Rev. A **77**, 030301 (2008).
 - [15] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
 - [16] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
 - [17] G. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (Cambridge Univ. Press, London, 1934).
 - [18] A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **87**, 040401 (2001).
 - [19] A. Kay, Phys. Rev. A **83**, 020303 (2011).
 - [20] O. Gühne, Phys. Lett. A **375**, 406 (2011).
 - [21] P. Horodecki, R. Horodecki, and M. Horodecki, Phys. Lett. A **232**, 333 (1997).